Elementary Particles of Conventional Field Theory as Regge Poles. III*

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It is shown that an elementary particle of conventional field theory may, under certain conditions, lie on a Regge trajectory. These conditions are that the system contain a "nonsense" channel at the angular momentum of the particle and that the Born approximation scattering amplitude factor in a well-defined way. They are satisfied by a spin $\frac{1}{2}$ fermion interacting through a conserved current with a spin one neutral boson. The particle in question is the fermion.

1. INTRODUCTION

IN early discussions of Regge poles it was assumed that an "elementary particle" of renormalized Lagrangian field theory would not lie on a Regge trajectory. Instead it would correspond to a special term in scattering amplitudes describing scattering at its particular value of J (angular momentum) and would not agree with the analytic continuation of the scattering amplitude from large ReJ. Such a term would contain a pole in the energy at the mass of the particle.

The belief that there was a sharp distinction between field theory and the Regge ideas was based on experience with field theories of scalar or pseudoscalar mesons interacting with spinor or scalar "nucleons." It was pointed out by Gell-Mann and Goldberger[1] that if one considers radiative corrections involving massive vector bosons the situation is quite different and that in fourth-order perturbation theory there is an indication that those special terms in the scattering amplitude which in second order look like fixed singularities in the angular momentum plane (like δ_{J0}, say) in fact get turned into moving Regge trajectories.

It is of course well known that Regge poles occur in the description of manifestly composite systems, both in potential theory and in simple approximations to field theory. It is a new and drastically different observation that an "elementary particle," appearing in the Lagrangian of a specific field theory, lies on a Regge trajectory.

We have found that a spin one-half elementary particle when coupled to a massive vector boson through a conserved current does, in fact, lie on a Regge trajectory. The same thing does *not* happen for a spin zero particle coupled to vector bosons. This is an exceedingly striking result. Although we can see how such a distinction arises between theories we have no deep explanation at present. It does show quite clearly at least two things: Not all field theories are the same and, contrary to what is frequently said, spin is an essential complication.

There are at present three principal approaches to relativistic quantum mechanics: axiomatic field theory, Lagrangian theory with Feynman diagrams, and dispersion and unitarity relations. No one has yet made any convincing argument that these approaches contradict one another; nor has anyone been able to make satisfactory calculations with any of them in the case of strong coupling. In particular, the contribution of axiomatic field theory to calculations has been less than any preassigned positive number, however small. The method of dispersion relations, sometimes referred to in the popular literature as "S-matrix theory," does not provide a complete calculational framework. The analyticity and generalized unitarity principles that are to provide the foundation of a dispersion theory are obtained in practice from experimentation with Feynman diagrams. It is a reasonable conjecture that these principles follow from the axioms of local field theory, which the Feynman diagrams formally obey.

The study of Feynman diagrams is a particularly useful tool for exploring the properties of relativistic quantum mechanics, since a direct appeal to the axioms has proved so difficult and tedious and since the "S-matrix theory" suffers from the disease of not yet existing. (One must admit, of course, that the sum of all the Feynman diagrams in a particular field theory may suffer from the disease of being wrong, even if it exists.)

A major new idea is the "bootstrap" hypothesis of Chew and collaborators, according to which all the strongly interacting particles are composite systems

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[1] M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962); erratum, Phys. Rev. Letters **10**, 39 (1963) (hereinafter referred to as I).

made up of one another, with dynamicably calculable ratios and coupling constants. This hypothesis has usually been stated in terms of dispersion theory, in the incomplete version now available. It is not clear, however, that the bootstrap idea is necessarily tied to that particular formulation of relativistic quantum mechanics.

It has also been suggested that the bootstrap hypothesis is equivalent to the statement that all strongly interacting particles lie on Regge trajectories. The fact that elementary particles in certain field theories are Reggeistic makes this equivalence unlikely (though not impossible) since it would demand that the bootstrap mechanism actually operate in some subtle way in conventional field theory to fix the coupling constant and mass ratio.

Apart from the strong interactions, there is the problem of quantum electrodynamics, where the bootstrap idea has not so far been applied. It is very interesting that the field theory in which the spinor particle if Reggeistic resembles so closely the only field theory in which we have any confidence. Of course our work applies just to the case in which the photon mass is nonvanishing, and the situation in real quantum electrodynamics remains to be investigated.

It is the purpose of the present paper and the succeeding one to amplify a brief discussion of the necessary and sufficient conditions for the Reggeization of an "elementary particle" published recently.[2] There it was shown that the critical feature of a particular theory was the presence or absence of a factoring property of the Born approximation and the presence of a "nonsense" channel (aside from certain questions involving subtractions in dispersion equations). We remarked that the factoring property obtained in a theory with spin one-half "nucleons" interacting with massive vector bosons, but failed for spin zero "nucleons." The latter point is treated in the following paper.[3] All of our earlier considerations have been confined to a study of perturbation theory in which we examine and sum the largest asymptotic term in a scattering amplitude for each power of the coupling constant. We continue this approximate discussion, in which the Regge angular momentum is calculated up to the first order in the coupling constant. Our trajectory thus includes only two-particle intermediate state contributions. This limitation enables us to use two-particle unitarity for analytically continued partial wave amplitudes.

The plan of the paper is as follows: In Sec. 2 a modification of the Jacob-Wick[4] theory of scattering is given which is very useful in Reggeization of scattering amplitudes.[5] It consists of using parity conserving helicity amplitudes and eliminates the d functions of Jacob and Wick in favor of Legendre functions. The general problem of Reggeization in the presence of spin is discussed in Sec. 3, and results for our special cases are given. In Appendix A general properties of the functions encountered in our formulation of scattering theory (definite combinations of Legendre functions) are given and a tabulation of all those encountered here and in any problem involving spins less than or equal to one. Details of the general Reggeization procedure are given in Appendix B where special attention is paid to the important concepts of "sense and nonsense" and compensating trajectories.[6] A criticism of the latter notion by Berestetsky[7] is disposed of at the same time. In Sec. 4 we treat the Compton scattering of massive vector bosons by spin one-half particles in Born approximation, showing the crucial factoring property and the appearance of a nonsense channel; the prediction of the answer to all orders is made on the basis of these results. The manner in which the unitarity and dispersion equations in the two-particle approximation show the necessity and sufficiency of the factoring property is taken up in Sec. 5. Unanswered questions involving subtraction constants are considered in Sec. 6 where we treat the complete fourth-order amplitude and show that our conjecture is exactly valid to that order. In addition, we show how the expected nth-order term may be extracted although we have not yet succeeded in eliminating all possible subtraction constants. The results are summarized in Sec. 7 and a number of theoretical implications and speculations are discussed.

2. PARITY-CONSERVING HELICITY AMPLITUDES

We treat collisions of the type $a+b \to c+d$, employing for the most part the notation of Jacob and Wick.[4] We note that on one of their helicity states, say $|JM;\lambda_c\lambda_d\rangle$, the parity operation P produces the effect

$$P|JM;\lambda_c\lambda_d\rangle = \eta_c\eta_d(-1)^{J-S_c-S_d}|JM;-\lambda_c-\lambda_d\rangle; \quad (2.1)$$

here λ means helicity, S spin, and η intrinsic parity. We want to define eigenstates $|JM;\lambda_c\lambda_d\rangle_\pm$ of parity such that

$$P|JM;\lambda_c\lambda_d\rangle_\pm = \pm(-1)^{J-v}|JM;\lambda_c\lambda_d\rangle_\pm, \quad (2.2)$$

where v is $\frac{1}{2}$ for half-integral J and 0 for integral J. A given Regge trajectory will then belong either to +

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[2] M. Gell-Mann, M. L. Goldberger, F. E. Low, and F. Zachariasen, Phys. Letters 4, 265 (1963) (hereinafter referred to as II).

[3] M. Gell-Mann, M. L. Goldberger, F. E. Low, V. Singh, and F. Zachariasen, Phys. Rev. 133, B161 (1964).

[4] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1954).

[5] Essentially the same method has been developed independently by F. Calogero, J. Charap, and E. Squires, Ann. Phys. (to be published).

[6] M. Gell-Mann, in *Proceeding of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962).

[7] V. Berestetsky, Phys. Letters 3, 175 (1963).

or to − when parity is conserved. We may take[8]

$$|JM;\lambda_c\lambda_d\rangle_\pm \equiv 2^{-1/2}|JM;\lambda_c\lambda_d\rangle \pm 2^{-1/2}\eta_c\eta_d(-1)^{S_c+S_d-v}|JM;-\lambda_c-\lambda_d\rangle. \quad (2.3)$$

Instead of the S matrix we use the F matrix defined by the relation

$$F_{fi}\equiv(S_{fi}-\delta_{fi})(2i)^{-1}k_F^{-1/2}k_i^{-1/2}, \quad (2.4)$$

where k refers to the center-of-mass momentum. When parity is conserved, F has no matrix elements between + states and − states and the nonvanishing matrix elements are given by the formula

$$\begin{aligned}F^J{}_{\pm\lambda_c\lambda_d;\,\lambda_a\lambda_b} &\equiv {}_\pm\langle JM;\lambda_c\lambda_d|F|JM;\lambda_a\lambda_b\rangle_\pm \\ &=\langle JM;\lambda_c\lambda_d|F|JM;\lambda_a\lambda_b\rangle \\ &\quad\pm\eta_c\eta_d(-1)^{S_c+S_d-v}\langle JM;-\lambda_c-\lambda_d|F|JM;\lambda_a\lambda_b\rangle \\ &=\langle\lambda_c\lambda_d|F^J|\lambda_a\lambda_b\rangle \\ &\quad\pm\eta_c\eta_d(-1)^{S_c+S_d-v}\langle-\lambda_c-\lambda_d|F^J|\lambda_a\lambda_b\rangle. \quad (2.5)\end{aligned}$$

The scattering amplitudes of Jacob and Wick, for azimuthal angle $\phi=0$, can be expressed as follows:

$$\begin{aligned}f_{\lambda_c\lambda_d;\,\lambda_a\lambda_b}(\theta) &= k_f^{1/2}k_i^{-1/2}\sum_J(2J+1)\langle\lambda_c\lambda_d|F^J|\lambda_a\lambda_b\rangle d_{\lambda\mu}{}^J(\theta); \\ &\qquad(\lambda=\lambda_a-\lambda_b,\,\mu=\lambda_c-\lambda_d). \quad (2.6)\end{aligned}$$

We now define parity-conserving scattering amplitudes by the rule

$$\begin{aligned}f^\pm{}_{\lambda_c\lambda_d;\,\lambda_a\lambda_b}(z) &\equiv[\sqrt2\cos(\theta/2)]^{-|\lambda+\mu|} \\ &\quad\times[\sqrt2\sin(\theta/2)]^{-|\lambda-\mu|}f_{\lambda_c\lambda_d\lambda_a\lambda_b}(\theta) \\ &\quad\pm(-1)^{\lambda+\lambda_m}\eta_c\eta_d(-1)^{S_c+S_d-v}[\sqrt2\sin(\theta/2)]^{-|\lambda+\mu|} \\ &\qquad\times[\sqrt2\cos(\theta/2)]^{-|\lambda-\mu|}f_{-\lambda_c,-\lambda_d;\,\lambda_a\lambda_b}(\theta), \quad (2.7)\end{aligned}$$

where $\lambda_m=\max(|\lambda|,|\mu|)$, $z=\cos\theta$. Correspondingly, we define new functions in place of the d's

$$\begin{aligned}e_{\lambda\mu}{}^{J\pm}(z) &\equiv 2^{-1}[\sqrt2\cos(\theta/2)]^{-|\lambda+\mu|} \\ &\quad\times[\sqrt2\sin(\theta/2)]^{-|\lambda-\mu|}d_{\lambda\mu}{}^J(\theta) \\ &\quad\pm(-1)^{\lambda+\lambda_m}2^{-1}[\sqrt2\sin(\theta/2)]^{-|\lambda+\mu|} \\ &\qquad\times[\sqrt2\cos(\theta/2)]^{-|\lambda-\mu|}d_{\lambda;\,-\mu}{}^J(\theta). \quad (2.8)\end{aligned}$$

Our final formula expressing scattering amplitudes in terms of F-matrix elements, with parity conservation, is then

$$\begin{aligned}f^\pm{}_{\lambda_c\lambda_k;\lambda_a\lambda_b} &= k_f^{1/2}k_i^{-1/2}\sum_J(2J+1)[e_{\lambda\mu}{}^{J+}(z)F^J{}_{\pm\lambda_e\lambda_d;\lambda_a\lambda_b} \\ &\qquad+e_{\lambda\mu}{}^{J-}(z)F^J{}_{\pm\lambda_e\lambda_d;\lambda_a\lambda_b}]. \quad (2.9)\end{aligned}$$

We see that $f^\pm$ has contributions from both $F^{J\pm}$ and $F^{J\mp}$, but when we Reggeize in the next section and consider large z, we will find that e^{J+} always dominates e^{J-} and the asymptotic behavior of $f^\pm$ is thus determined by $F^{J\pm}$.

[8] Note that, for both helicities equal to zero and $\eta_c\eta_d(-1)^{S_c+S_d}=1$, the minus-state vanishes while the plus-state vector has length $\sqrt2$ instead of 1. The matrix elements $F_{\lambda_c\lambda_d;\lambda_a\lambda_b}{}^{J\pm}$ of Eq. (2.5) reflect this peculiarity, which must be taken into account in unitary equations when either pair of λ's vanishes. For $\eta_c\eta_d(-1)^{S_c+S_d}=-1$, the plus-state vanishes.

To invert the partial wave expansion (2.9) we define

$$\begin{aligned}2c_{\lambda\mu}{}^{J\pm} &\equiv[\sqrt2\cos(\theta/2)]^{|\lambda+\mu|}[\sqrt2\sin(\theta/2)]^{|\lambda-\mu|}d_{\lambda\mu}{}^J \\ &\quad\pm(-1)^{\lambda+\lambda_m-1}[\sqrt2\sin(\theta/2)]^{|\lambda+\mu|} \\ &\qquad\times[\sqrt2\cos(\theta/2)]^{|\lambda-\mu|}d_{\lambda,-\mu}{}^J. \quad (2.10)\end{aligned}$$

We then obtain the inversion formula

$$\begin{aligned}F^J{}_{\pm\lambda_c\lambda_d;\lambda_a\lambda_b} &= 2^{-1}k_f^{-1/2}k_i^{1/2}\int_{-1}^{1}dz[c_{\lambda\mu}{}^{J+}(z)f^\pm{}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z) \\ &\qquad+c_{\lambda\mu}{}^{J-}(z)f^\mp{}_{\lambda_c\lambda_d;\lambda_a\lambda_b}(z)]. \quad (2.11)\end{aligned}$$

The properties of the c's and e's are described in Appendix A. The e's are given in terms of a simple linear operator applied to $P_{J\pm v}$, while the c's can be written as linear combinations of Legendre functions $P_{J-\lambda_m}\cdots P_{J+\lambda_m}$ with constant coefficients. We have tabulated the functions for non-negative λ and μ up to 2, with general J.

When λ or $\mu>J$ and $J+v$ is integral, then that value of J is not reached physically and we refer to the channel as "nonsense" for the particular J under consideration. Otherwise the channel corresponds to "sense."[6]

Let us now specialize to the case of elastic scattering of a vector particle by a spinor particle. We have $S_a=S_c=1$, $S_b=S_d=1/2$, $\eta_a=\eta_c=-1$, $\eta_b=\eta_d=+1$, $v=1/2$, $k_i=k_f=k$. We define $l=J-1/2$. To avoid duplicating amplitudes $f^\pm$ and $F^\pm$, we restrict their indices λ_b and λ_d to the value $+1/2$ and we then supress those indices entirely, writing simply $F_{\lambda_c\lambda_a}{}^l(z)$ with λ_a and $\lambda_c=-1$, 0, and 1. For each parity there are thus six distinct elements of the symmetric matrix F^l. Since J is half-integral, the matrix elements of opposite parity are related by MacDowell's formula[9]

$$F_{\lambda_c\lambda_a}{}^{l\pm}(W)=-F_{\lambda_c\lambda_a}{}^{l\mp}(-W), \quad (2.12)$$

where W is the total energy. Thus we need concern ourselves only with one sign of the F's, which we take to be +.

The nucleon as an intermediate state represents a pole in the F^+ matrix at $J=1/2$ (or $l=0$) and $W=m$.

At $l=0$, the channels with λ_a or $\lambda_c=0$, 1 are sensible while that with λ_a or $\lambda_c=-1$ is nonsensical. We let the Greek indices κ, ν, etc., run over the sensible values 0, 1. It turns out that the partial wave expansions (2.9) for our problem can be written as follows, with the aid of the formulas in Appendix A:

$$\begin{aligned}\frac{f_{-1,-1}{}^\pm}{\sqrt2} &= \sum_l\frac{P_{l+1}''+zP_{l+1}'''+P_l'''}{l(l+2)}F_{-1,-1}{}^{l\pm} \\ &\quad-\sum_l\frac{P_{-l+1}'''+zP_l'''+P_l''}{l(l+2)}F_{-1,-1}{}^{l\mp}, \quad (2.13)\end{aligned}$$

[9] S. MacDowell, Phys. Rev. **116**, 774 (1959).

$$\frac{f_{-1,\nu}{}^{\pm}}{\sqrt{2}}=\sum_{l}\frac{P_{l+1}{}''F_{-1,\nu}{}^{l\pm}}{[l(l+2)]^{1/2}}-(-1)^{\nu}\sum_{l}\frac{P_{l}{}''F_{-1,\nu}{}^{l\mp}}{[l(l+2)]^{1/2}}, \quad (2.14)$$

$$f_{\kappa\nu}{}^{\pm}/\sqrt{2}=\sum_{l}P_{l+1}{}'\epsilon_{\nu\kappa}F_{\kappa\nu}{}^{l\pm}-\sum_{l}P_{l}{}'\epsilon_{\kappa\nu}F_{\kappa\nu}{}^{l\mp}, \quad (2.15)$$

where $\epsilon_{\kappa\nu}=1$ unless $\kappa=0$, $\nu=1$, in which case $\epsilon_{\kappa\nu}=-1$. Similarly, the inversion formulas (2.11) for integral l become

$$\sqrt{2}F_{-1,-1}{}^{l\pm}=\frac{1}{2}\int_{-1}^{1}dz\left[\frac{(l+1)P_{l-1}+3lP_{l+1}}{2l+1}f_{-1,-1}{}^{\pm}+\frac{3(l+2)P_{l}+lP_{l+2}}{2l+3}f_{-1,-1}{}^{\mp}\right], \quad (2.16)$$

$$\frac{\sqrt{2}F_{-1,\nu}{}^{l\pm}}{[l(l+2)]^{1/2}}=\frac{1}{2}\int_{-1}^{1}dz\left[\frac{P_{l-1}-P_{l+1}}{2l+1}f_{-1,\nu}{}^{\pm}+(-1)^{\nu}\frac{P_{l}-P_{l+2}}{2l+3}f_{-1,\nu}{}^{\mp}\right], \quad (2.17)$$

$$\sqrt{2}F_{\kappa\nu}{}^{l\pm}=\frac{1}{2}\int_{-1}^{1}dz[P_{l}\epsilon_{\nu\kappa}f_{\kappa\nu}{}^{\pm}+P_{l+1}\epsilon_{\kappa\nu}f_{\kappa\nu}{}^{\mp}]. \quad (2.18)$$

3. REGGE POLE CONTRIBUTIONS

Now let us consider what happens when we "Reggeize" the partial wave formulas. We treat the contribution of a moving pole in the l plane to the scattering amplitudes, when the partial wave sums are replaced by Sommerfeld-Watson integrals.

In Appendix B we discuss the general question of Reggeizing in the presence of spin, particularly in connection with "sense" and "nonsense" and with twin trajectories. Here we merely apply the results to our problem of elastic vector-spinor scattering. We consider a moving pole in the F^{l+} matrix elements near $l=0$, with the idea of relating this trajectory to the nucleon. For the time being, for simplicity, we ignore exchange forces and signature.

Let the position of the Regge pole be given by $l=\alpha(W)$. Then the partial wave amplitudes will have the form

$$\begin{aligned}F_{\kappa\nu}{}^{l+}&\approx\eta_{\kappa}(W)\eta_{\nu}(W)/[l-\alpha(W)],\\ F_{-1,\nu}{}^{l+}/[l(l+2)]^{1/2}&\approx\zeta_{-1}(W)\eta_{\nu}(W)/[l-\alpha(W)], \quad (3.1)\\ F_{-1,-1}{}^{l+}&\approx[\zeta_{-1}(W)]^{2}\alpha(W)[\alpha(W)+2]/[l-\alpha(W)],\end{aligned}$$

in the neighborhood of the pole. Here we have used the factorization property of the residues of Regge poles.[10] If the trajectory chooses sense at $\alpha=0$, so as to produce a physical nucleon, then the amplitudes η_ν approach finite numbers as $\alpha\to0$. The amplitude $\zeta_{-1}[\alpha(\alpha+2)]^{1/2}$ for the nonsense channel vanishes like $\alpha^{1/2}$ as $\alpha\to0$, so that ζ_{-1} also approaches a finite number as $\alpha\to0$. We have written $F_{-1,\nu}{}^{+}$ divided by $[l(l+2)]^{1/2}$ so as to treat a quantity without fixed branch points in the l plane at $l=0$ and $l=-2$.

If the trajectory were to choose nonsense at $\alpha=0$, then we would have, instead of (3.1), the relations

$$\begin{aligned}F_{\kappa\nu}{}^{l+}&\approx\zeta_{\kappa}\zeta_{\nu}\alpha(\alpha+2)/(l-\alpha),\\ F_{-1,\nu}{}^{l+}/[l(l+2)]^{1/2}&\approx\eta_{-1}\zeta_{\nu}/(l-\alpha), \quad (3.2)\\ F_{-1,-1}{}^{l+}&\approx[\eta_{-1}]^{2}/(l-\alpha)\end{aligned}$$

with ζ and η again approaching finite numbers as $\alpha\to0$.

Returning to (3.1), we now examine the asymptotic form of the partial wave expansions [(2.13)–(2.15)] at large z, Reggeizing as in Appendix B. We obtain the results

$$\begin{aligned}f_{-1,-1}{}^{+}&\to N_{\alpha}\zeta_{-1}{}^{2}\pi(\sin\pi\alpha)^{-1}\alpha^{2}(\alpha+1)(-z)^{\alpha-1},\\ f_{-1,\nu}{}^{+}&\to N_{\alpha}\zeta_{-1}\eta_{\nu}\pi(\sin\pi\alpha)^{-1}\alpha(\alpha+1)(-z)^{\alpha-1}, \quad (3.3)\\ f_{\kappa\nu}{}^{+}&\to -N_{\alpha}\epsilon_{\nu\kappa}\eta_{\kappa}\eta_{\nu}\pi(\sin\pi\alpha)^{-1}(\alpha+1)(-z)^{\alpha},\end{aligned}$$

where $N_\alpha=\sqrt{2}2^{\alpha+1}\Gamma(\alpha+3/2)[\sqrt{\pi}\Gamma(\alpha+2)]^{-1}$. Near $\alpha=0$, the sense-sense amplitudes $f_{\kappa\nu}{}^{+}$ go like $(-z)^0$ at large z, while the others go like $(-z)^{-1}$. In fact we have, as $\alpha\to0$, $z\to\infty$,

$$\begin{aligned}f_{-1,-1}{}^{+}&\to-\sqrt{2}\zeta_{-1}{}^{2}\alpha z^{-1},\\ f_{-1,\nu}{}^{+}&\to-\sqrt{2}\zeta_{-1}\eta_{\nu}z^{-1}, \quad (3.4)\\ f_{\kappa\nu}{}^{+}&\to\sqrt{2}\epsilon_{\nu\kappa}\eta_{\kappa}\eta_{\nu}\alpha^{-1}.\end{aligned}$$

In terms of the quantities occurring here, we may rewrite the expressions (3.1) for the partial wave amplitudes near the pole $\alpha\to0$. Using the fact that $\alpha(l-\alpha)^{-1}\to-\delta_{0l}$ as $\alpha\to0$, we obtain for the pole contributions near $\alpha=0$ the following:

$$\begin{aligned}F_{\kappa\nu}{}^{l+}&\approx-(\eta_{\kappa}\eta_{\nu}\alpha^{-1})\delta_{0l},\\ F_{-1,\nu}{}^{l+}/[l(l+2)]^{1/2}&\approx\zeta_{-1}\eta_{\nu}l^{-1}, \quad (3.5)\\ F_{-1,-1}{}^{+}&\approx2(\zeta_{-1}{}^{2}\alpha)l^{-1},\end{aligned}$$

where $\eta_\kappa\eta_\nu\alpha^{-1}$ goes like α^{-1} near $\alpha=0$, $\zeta_{-1}\eta_\nu$ stays finite, and $\zeta_{-1}{}^2\alpha$ vanishes like α.

For general l, the definitions of $F_{\lambda_c\lambda_a}{}^{\pm}$ are not, of course, exactly the same as the inversion formulas [(2.16)–(2.18)], which are valid for physical l. Instead, we must normally employ the Froissart definition, which coincides with (2.16)–(2.18) at physical l. For the Froissart definition, each f is replaced by its weight function w in a dispersion relation in z, each P_n is replaced by $2Q_n$, and the integration over z is over the region in which $w\neq0$. Keeping only the terms that are important for $l\approx\alpha\approx0$, we obtain

$$\begin{aligned}\sqrt{2}F_{\kappa\nu}{}^{l+}&\approx\int dzQ_{l}(z)\epsilon_{\nu\kappa}w_{\kappa\nu}{}^{+}(z),\\ \sqrt{2}F_{-1,\nu}{}^{l+}/[l(l+2)]^{1/2}&\approx\int dzQ_{l-1}(z)w_{-1,\nu}{}^{+}(z), \quad (3.6)\\ \sqrt{2}F_{-1,-1}{}^{l+}&\approx\int dz2Q_{l-1}(z)w_{-1,-1}{}^{+}(z).\end{aligned}$$

[10] M. Gell-Mann, Phys. Rev. Letters 8, 263 (1962); V. Gribov and I. Pomeranchuk, Phys. Rev. Letters 8, 343 (1962).

In an approximation in which α is treated as very small, the last two of these equations can lead to the results of Eq. (3.5), since Q_{l-1} has a pole in l at $l=0$. We must not, however, expect to pick up the Kronecker delta for $F_{\kappa\nu}{}^{l+}$ from the Froissart definition in this limiting approximation. Instead, we see from (3.4) that $f_{\kappa\nu}{}^{+}$ has a term in z^0 as $\alpha \to 0$, with no weight function, and the Kronecker delta will show up in the ordinary inversion formula (2.18).

In an exact treatment, the Froissart definitions are all right, if they are used at large Rel and analytically continued, and they should lead to the formulas (3.1) for the behavior of $F_{\lambda_a\lambda_c}{}^{+}$ near the Regge pole. We have mentioned the approximation of very small α because it corresponds to the Born approximation in field theory, which we discuss in the next two sections.

4. CALCULATION OF SECOND ORDER AND FORMULATION OF *n*th ORDER

We consider the interaction through a conserved current of a spin 1/2 fermion of mass m (which we call the nucleon) with a spin 1 boson of mass λ (which we call the photon). The initial and final four-momenta of the nucleon are p_1 and p_2, respectively, and those of the photon are k_1 and k_2. The nucleon energy is called E, and that of the photon ω. Boldface **p** and **k** stand for three vectors. Solid lines in Feynman diagrams represent nucleons, dotted lines photons and wavy lines spinless mesons. We employ γ's and a metric such that the Dirac equation is $i\gamma\cdot p+m=0$ and such that $p^2+m^2=0$.

The Compton scattering Feynman amplitude M and the conventional scattering amplitude f are given by

$$f=(m/4\pi W)M\,,\quad M=\bar{u}_2\mathfrak{M}_{\mu\nu}u_1\epsilon_{2\mu}\epsilon_{1\nu} \qquad (4.1)$$

where $\bar{u}u=1$ and ϵ_1 and ϵ_2 are the initial and final photon polarization four vectors, normalized to unity. We have $\epsilon_1 k_1=\epsilon_2 k_2=0$. W is the total c.m. energy. The conservation of current, expressed by the equations

$$k_{2\mu}\mathfrak{M}_{\mu\nu}=0\,, \qquad (4.2)$$

$$\mathfrak{M}_{\mu\nu}k_{1\nu}=0\,, \qquad (4.3)$$

then allows us to ignore temporal components of the ϵ's and modify the longitudinal components accordingly.

Equations (4.2) and (4.3) also permit us to change the gauge of external photons according to the rule

$$\epsilon_\mu \to \epsilon_\mu - \epsilon\cdot A k_\mu\,, \qquad (4.4)$$

where the vector A is completely arbitrary.

Our first step is to calculate the second-order Feynman amplitude for the process $\gamma+N\to\gamma+N$. The relevant diagrams are shown in Fig. 1.

$$\mathfrak{M}_{\mu\nu}=\gamma^2[i\Gamma_\mu[1/(i\gamma\cdot p+m)]i\Gamma_\nu + i\Gamma_\nu(1/i\gamma\cdot q+m)i\Gamma_\mu]\,, \qquad (4.5)$$

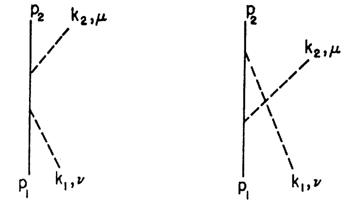

FIG. 1. Born approximation diagram for Compton scattering.

where $p=p_1+k_1$, $q=p_1-k_2$, γ is the coupling constant and the gauges

$$\Gamma_\mu=\gamma_\mu-(\gamma\cdot k_2 k_{1\mu}/k_1\cdot k_2)\,,\ \Gamma_\nu=\gamma_\nu-(\gamma\cdot k_1 k_{2\nu}/k_1\cdot k_2) \qquad (4.6)$$

have been chosen for the initial and final photons, respectively. The point of this choice is that as $\cos\theta\to\infty$, all the matrix elements of Γ_μ and $\Gamma_\mu\to$ constant, as do the anticommutators $\{\Gamma_\mu,\gamma\cdot q\}$, where q is any momentum in the problem. Therefore, in this gauge, the crossed diagram contributes only to order $1/z$ compared with the uncrossed diagram. Since all our calculations will be asymptotic in z, we may thus restrict our attention to uncrossed diagrams. The simplification thus attained, although modest in second order, is enormous in fourth and higher orders.

Following Jacob and Wick, we take the incident photon in the $+z$ direction, and therefore the target nucleon in the $-z$ direction in the c.m. system. The final photon momentum is in the $x-z$ plane, and makes an angle θ with the z axis.

The spinors describing the initial and final nucleons, respectively, are

$$u_1=\left(1+\frac{2k\lambda_b\rho_1}{m+E}\right)\chi_{-\lambda_b}\left(\frac{m+E}{2m}\right)^{1/2}$$

and

$$u_2=\left(1+\frac{2k\lambda_d\rho_1}{m+E}\right)\left(\frac{1-2\lambda_d\boldsymbol{\sigma}\cdot\hat{k}_2}{2\cos(\theta/2)}\right)\chi_{-\lambda_d}\left(\frac{m+E}{2m}\right)^{1/2}, \qquad (4.7)$$

where λ_b and λ_d are the initial and final nucleon helicities, respectively, and χ_λ is a spin state with $\sigma_z=2\lambda$.

The photon polarizations are given as follows (with temporal components of ϵ's eliminated, as described above):

$$\lambda_a=\pm1:\ \boldsymbol{\varepsilon}_1=-\lambda_a(\hat{\imath}+i\lambda_a\hat{\jmath})/\sqrt{2}\,,$$

$$\lambda_a=0:\quad \boldsymbol{\varepsilon}_1=\hat{k}_1\lambda/\omega\,.$$

$$\lambda_c=\pm1:\ \boldsymbol{\varepsilon}_2=-\lambda_c(\hat{\imath}\cos\theta-\hat{k}\sin\theta-i\lambda_c\hat{\jmath})/\sqrt{2}\,,$$

$$\lambda_c=0:\quad \boldsymbol{\varepsilon}_2=\hat{k}_2\lambda/\omega\,.$$

The scattering amplitudes may now be calculated in the high z limit. As in previous sections we specify states only by photon helicity: thus $f^{+}{}_{\lambda_c\lambda_d,\lambda_a\lambda_b}\to f^{+}{}_{\lambda_c\lambda_a}$,

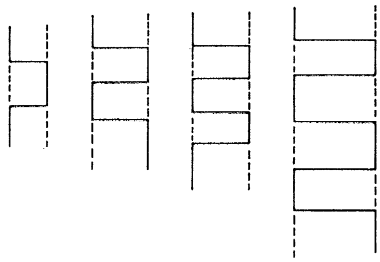

FIG. 2. Higher-order ladder diagrams which generate the nucleon Regge pole.

etc. We find

$$
\begin{aligned}
f_{11}{}^{+}&=-\frac{\gamma^2}{\sqrt{2}}\frac{E+m}{8\pi Wk^2}\frac{(E-m-\omega)^2}{W-m}+O\left(\frac{1}{z}\right),\\
f_{00}{}^{+}&=-\frac{\gamma^2}{\sqrt{2}}\frac{E+m}{4\pi Wk^2}\frac{\lambda^2}{W-m}+O\left(\frac{1}{z}\right),\\
f_{01}{}^{+}&=\frac{\gamma^2}{\sqrt{2}}\frac{E+m}{4\pi\sqrt{2}Wk^2}\frac{\lambda(E-m-\omega)}{W-m}+O\left(\frac{1}{z}\right),\\
f_{-1-1}{}^{+}&=-\frac{\gamma^2}{\sqrt{2}}\frac{E+m}{8\pi Wk^2}(W-m)\frac{1}{z}+\left(\frac{1}{z^2}\right),\\
f_{-10}{}^{+}&=-\frac{\gamma^2}{\sqrt{2}}\frac{E+m}{4\pi\sqrt{2}Wk^2}\cdot\lambda\cdot\frac{1}{z}+O\left(\frac{1}{z^2}\right),\\
f_{-11}{}^{+}&=\frac{\gamma^2}{\sqrt{2}}\cdot\frac{E+m}{8\pi Wk^2}(E-\omega-m)\cdot\frac{1}{z}+O\left(\frac{1}{z^2}\right).
\end{aligned}
\tag{4.8}
$$

Comparing with Eq. (3.4), we see that the Born approximation at large z corresponds exactly to the contribution at large z of a Regge pole with $\alpha\rightarrow 0$ as $\gamma^2\rightarrow 0$. We discover that the leading term in α is of order γ^2; it is then clear that the leading terms in $\eta_\mu{}^2$ and $\zeta_{-1}{}^2$ are of order γ^4 and 1, respectively:

$$
\begin{aligned}
\xi_{-1}&\equiv\sqrt{2}\alpha^{1/2}\zeta_{-1}=[\gamma^2(E+m)(W-m)/8\pi Wk^2]^{1/2},\\
\xi_0&\equiv\alpha^{-1/2}\eta_0=\{[\gamma^2(E+m)/8\pi Wk^2]\cdot[\lambda^2/(W-m)]\}^{1/2},\\
\xi_1&\equiv\alpha^{-1/2}\eta_1=-\{[\gamma^2(E+m)/16\pi Wk^2]\\
&\qquad\cdot[(E-m-\omega)^2/W-m]\}^{1/2}.
\end{aligned}
\tag{4.9}
$$

The quantities ξ give a very simple form to the expressions (3.5) for the partial wave amplitudes. We see that for $\alpha\rightarrow 0$ at $W=m$ like $W-m$ (that is, a physical nucleon lying on the Regge trajectory), the Born approximation formulas for the ξ's have the right behavior, with ξ_{-1} going like $(W-m)^{1/2}$ and ξ_0 and ξ_1 like $(W-m)^{-1/2}$.

If the Regge behavior persists in higher order, then the asymptotic forms of the f^+ amplitudes will go as in Eq. (3.3). Keeping just the highest power of $\ln(-z)$ for each power of γ^2, we find that each f^+ is just multiplied by $(-z)^{\alpha(W)}=\exp[\alpha(W)\ln(-z)]$.

It can be shown that for the asymptotic f^+ amplitudes the Dirac matrix $i\gamma\cdot p$ occurring between Γ_μ and Γ_ν acts like $-W$. Therefore, if we want to multiply the asymptotic f^+ by a function of W, we can do that by placing the same function of $-i\gamma\cdot p$ between Γ_μ and Γ_ν.

The behavior of higher order Feynman diagrams is now clear if the Regge pole behavior persists. We must have, to all orders,

$$
\mathfrak{M}_{\mu\nu}\underset{z\rightarrow\infty}{\longrightarrow}\gamma^2 i\Gamma_\mu[(-z)^{\alpha(-i\gamma\cdot p)}/(i\gamma\cdot p+m)]i\Gamma_\nu,\tag{4.10}
$$

where we retain the highest power of $\ln(-z)$ for each power of γ^2 and use just the second-order α. It is this behavior that we must try to establish.

One amendment is still in order. We show later that the leading terms come from the set of graphs shown in Fig. 2. These have alternately t cuts in even orders of γ^2 and u cuts in odd orders, whereas $(-z)^\alpha$ has only t cuts in all orders.[11] This necessitates the introduction of two trajectories, one of positive signature, equal to α, and the second of negative signature, equal to $-\alpha$, so that $(-z)^\alpha$ is replaced by

$$
\left[\frac{(-z)^\alpha+z^\alpha}{2}-\frac{(-z)^{-\alpha}-z^{-\alpha}}{2}\right]
=1+\alpha\ln(-z)+\frac{\alpha^2}{2!}(\ln z)^2+\cdots
$$

which has its cuts appropriately placed.

5. ANGULAR MOMENTUM BEHAVIOR

The singularities in the partial wave amplitudes at $l=0$ in Born approximation should now be given by the approximation of Eq. (3.5), that is, small α and small l. In terms of the quantities ξ defined and evaluated in lowest order in Eq. (4.9), we obtain

$$
\begin{aligned}
F_{\kappa\nu}{}^{l+}&=-\xi_\kappa\xi_\nu\delta_{0l},\\
F_{-1\nu}{}^{l+}/l^{1/2}&=\xi_{-1}\xi_\nu(1/l),\\
F_{-1-1}{}^{l+}&=(\xi_{-1})^2(1/l).
\end{aligned}
\tag{5.1}
$$

Alternatively, we may take the explicit forms of the amplitudes f in Born approximation and compute the singularities in the F^+ at $l=0$ using normal inversion formulas (2.18) for $F_{\kappa\nu}$ and Froissart definitions (3.6) otherwise. The result is the same, and again we obtain the expressions (4.9) for the ξ's.

The appearance of the $1/l$ terms is critical to the success of our program, and is characteristic of the

[11] In this paper we consider scattering in the s channel, where s is the square of the energy in the c.m. system. Large $z=\cos\theta$ then corresponds to large $t=-2k^2(1-z)$, where k is the c.m. momentum. The third Mandelstam variable is $u=2\lambda^2+2m^2-s-t$. The present convention is different from that of I and II, in which the scattering was in the u channel, t was the momentum transfer, and s the crossed momentum transfer. The problem was treated in the u channel in I and II since for large t and s and finite negative u we were in the physical region in the s channel: high-energy and backward scattering. In the present paper our use of unitarity in the s channel requires us to use $s>(m+\lambda)^2$, so that we are in any case not in a physical region for $z\rightarrow\infty$. We have therefore reverted to the more usual notation.

existence of a "nonsense" channel, to wit the state we have called -1: $\lambda_a=-1$, $\lambda_b=1/2$, $M=J_z=-3/2$ and therefore $J=1/2$ unphysical, or nonsense. The iteration of $1/l$ through unitarity and dispersion relations clearly can produce a sequence like

$$\frac{1}{l}+\frac{\alpha}{l^2}+\frac{\alpha^2}{l^3}+\cdots=\frac{1}{l-\alpha},$$

whereas the iteration of δ_{l0} produces nothing but δ_{l0}. The presence of such a nonsense channel is a consequence of the nonzero spin of the photon, and shows the necessity for introducing spin 1 particles into a field theory in which an "elementary" particle is to be transformed into a Regge pole.

The δ_{l0} terms come from channels which are physical (or sensible) at $l=0$, and therefore have the nucleon pole at $W=m$. The effect of higher order corrections on these terms must be to replace δ_{l0} by $-\alpha/(l-\alpha)$.

The second crucial property for Reggeization is also evident from inspection of the coefficients of δ_{l0}, $1/l^{1/2}$, and $1/l$, respectively, in the F's in Born approximation. It is that they factorize into products of ξ's, as in (5.1). We have no deep explanation for the factorization of the Born approximation. It is an important and hitherto undiscovered property of conventional field theory which makes Regge pole behavior possible. It is not true of all field theories that contain a nonsense channel; for example, it is not true for spin zero nucleons interacting with massive photons. At present we must consider it to be a remarkable accident, which selects the particular field theory we have been considering.

To illustrate the mechanism for the generation of the Regge pole we consider as an example the case of scalar nucleon scattering with exchange of scalar mesons. The pole in question starts at $l=-1$ instead of $l=0$. The invariant scattering amplitude M is given in lowest order by

$$M=g^2/(\mu^2-t) \tag{5.2}$$

or

$$\mathfrak{M}_l=\frac{1}{2}\int_{-1}^{1}P_l(z)M(s,z)dz,$$

$$\to g^2/[2p^2(l+1)]+\text{terms regular at } l=-1.$$

The unitarity equation is

$$\mathrm{Im}\mathfrak{M}_l=(p/8\pi W)|\mathfrak{M}_l|^2$$

or (5.3)

$$\mathrm{Im}(p^2\mathfrak{M}_l)=(1/8\pi pW)|p^2\mathfrak{M}_l|.$$

The dispersion relation satisfied by $t_l\equiv p^2\mathfrak{M}_l$, neglecting inelasticity and left-hand cuts, is

$$t_l=\frac{g^2}{2(l+1)}+\frac{1}{\pi}\int\frac{ds'}{s'-s}\mathrm{Im}t_l, \tag{5.4}$$

of which the power series solution is

$$t_l=\frac{g^2}{2(l+1)}\bigg/\left(1-\frac{g^2}{16\pi^2}\cdot\int\frac{ds'}{p'W'}\cdot\frac{1}{s'-s}\cdot\frac{1}{l+1}\right)$$

or

$$\mathfrak{M}_l=\frac{g^2}{2p^2(l-\alpha)},$$

with

$$\alpha=-1+\frac{g^2}{16\pi^2}\int\frac{ds'}{p'W'(s'-s)}. \tag{5.5}$$

In our case, the generation of the Regge pole near $l=0$ works in a very similar way.

The unitarity equation for $F_{-1,-1}$ is

$$\mathrm{Im}F_{-1,-1}=k\textstyle\sum_\lambda|F_{-1,\lambda}|^2. \tag{5.6}$$

In the neighborhood of $l=0$, $\lambda=-1$ contributes the most singular term. The ansatz

$$F_{-1,-1}=\xi_{-1}^2/l-\alpha, \tag{5.7}$$

which is suggested by the Born approximation (5.1), is clearly consistent with the unitarity equation thus obtained, provided

$$\mathrm{Im}\alpha=k\xi_{-1}^2=\gamma^2\frac{(E+m)(W-m)}{8\pi kW}. \tag{5.8}$$

This agrees with the value for α found in Ref. 1 (Erratum):

$$\alpha=\gamma^2[(W-m)/8\pi^2][(W+m)I_0-WI_1], \tag{5.9}$$

where

$$I_n=\int_0^1 dxx^n/[m^2x+\lambda^2(1-x)-sx(1-x)] \tag{5.10}$$

or

$$I_0=\int_{(m+\lambda)^2}^{\infty}\frac{ds'}{k'W'(s'-s)},$$

and

$$I_1=\int_{(m+\lambda)^2}^{\infty}\frac{ds'\omega'}{k's'(s'-s)}. \tag{5.11}$$

We may also write α as in Ref. 2:

$$\alpha(W)=\gamma^2\frac{(W-m)}{8\pi^2}\int_{m+\lambda}^{\infty}\frac{dW'}{k'W'}\times\left(\frac{E'+m}{W'-W-i\epsilon}-\frac{E'-m}{W'+W+i\epsilon}\right). \tag{5.12}$$

Our argument shows that if our assumption for $F_{-1,-1}$ is correct in the nth order of γ^2 it will be correct for the imaginary part in $n+1$st order of γ^2. Also, the form (5.6) for $F_{-1,-1}$ satisfies the usual analyticity requirements. However, the complete verification of its form depends on a precise knowledge of the nature of the subtractions in partial wave dispersion relations, which we have not as yet been able to obtain. Lacking

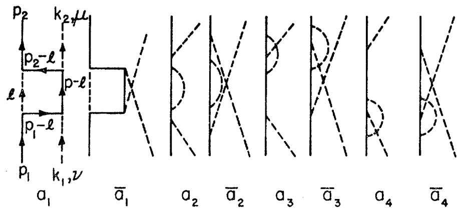

FIG. 3. Fourth-order Compton scattering diagrams.

it, we have been forced to turn to experiment, i.e., fourth and higher order perturbation theory, which we describe in the next section.

An identical calculation can be performed for $F_{-1,\mu}$ where $\mu=0$ or 1. Again the dominant intermediate state is -1, so that the unitarity relation reads

$$\mathrm{Im}F_{-1,\mu}=kF_{-1,-1}{}^{*}F_{-1,\mu} \tag{5.13}$$

which is satisfied by the new ansatz

$$F_{-1,\mu}=\xi_{-1}\xi_{\mu}l^{1/2}/(l-\alpha) \tag{5.14}$$

with real ξ_μ together with the old assumption for $F_{-1,-1}$.

Finally, the approximate unitarity equation for F_μ ($\mu,=0$, 1) is

$$\mathrm{Im}F_{\mu\nu}=kF_{\mu,-1}{}^{*}F_{\nu,-1}\,, \tag{5.15}$$

which is solved by setting

$$F_{\mu\nu}=\xi_\mu\xi_\nu[\alpha/(l-\alpha)]\,. \tag{5.16}$$

The agreement of our formulas for $F_{-1,-1}$ and $F_{-1,\mu}$ with the calculated Born approximation does not depend on factorization, since the scale of $F_{-1,\mu}$ at each energy is not determined by unitarity [Eq. (5.13)] and can be arbitrarily chosen to fit the lowest order of perturbation theory. Once ξ_μ has been chosen in this way, however, $F_{\mu\nu}$ is determined in 4th and higher order, as shown by Eq. (5.15). The second-order limit in (5.15) is $-\xi_\mu\xi_\nu\delta_{l0}$, which need not agree with the second-order calculation of a specific theory (e.g., scalar nucleons+vector mesons).[3] As shown by Eq. (5.1), it does agree for the case treated in this article.

Our cavalier treatment of left-hand cuts can be corrected by using the method of asymptotic unitarity for the f amplitudes, developed in the succeeding paper[3] for the scalar-vector scattering. The unitarity arguments are presented there using f amplitudes at large z instead of amplitudes near $l=0$. It is still true, however, that in each order we must resort to a "measurement" of Feynman diagrams in order to show the absence of subtraction constants.

6. FOURTH- AND nth-ORDER PERTURBATION THEORY

In this section we carry out a direct "measurement" of the Compton amplitude in the limit of large $\cos\theta$. In this way we compute $\alpha(W)$ directly and verify explicitly that at least to fourth order in γ the assumptions about subtraction constants made in Sec. 5 are borne out. As a matter of fact we exhibit in arbitrary order precisely the term which corresponds to our complete prediction of the large $\cos\theta$ amplitude given previously:

$$\mathfrak{M}\approx\gamma^2 i\Gamma_\mu\frac{(-z)^{\alpha(-i\gamma\cdot p)}}{i\gamma\cdot p+m}i\Gamma_\nu+\text{signature terms}. \tag{6.1}$$

We begin by looking at fourth-order perturbation theory and expect according to the above formula to find a term proportional to $\ln(-z)\approx\ln(-t)$, where $(-t)$ is the square of the momentum transfer, $-t=(p_1-p_2)^2$. There have been a number of papers on the subject of extracting the high-energy behavior of Feynman diagrams.[12] Whereas we rely partly on this work it is instructive to consider the fourth-order problem explicitly if for no other reason than completeness. Actually, the more elaborate treatments have been confined to the somewhat simpler case of spinless particles whereas in our case spin considerations are crucial.

For orientation and as a guide to the natural order of magnitude of the integrals we encounter, let us begin by writing down the result we would obtain for the diagram a_1 of Fig. 3 if all the particles were spinless. This diagram contributes the leading term in $\ln(-t)$, even in the case of particles with spin, provided the gauge Γ_μ is used, since all other uncrossed diagrams are essentially independent of t, just as in the Born approximation. In such a case we have for the Feynman matrix element, called $\mathfrak{M}_0$,

$$\mathfrak{M}_0=\int\frac{d^4l}{(2\pi)^4 i}\,\frac{1}{[(p_1-l)^2+m^2][p_2-l)^2+m^2][p-l)^2+m^2][l^2+\lambda^2]}\,, \tag{6.2}$$

and with the standard parameterization and integration over l this becomes (recall $p^2=-s$)

$$\begin{aligned}\mathfrak{M}_0&=\frac{1}{16\pi^2}\int\frac{dx_1\cdots dx_4\delta(1-x_1-x_2-x_3-x_4)}{[\lambda^2x_4+(m^2-s)x_3-(p_1x_1+p_2x_2+px_3)^2]^2}\,,\\&=\frac{1}{16\pi^2}\int\frac{dx_1\cdots dx_4\delta(1-x_1-x_2-x_3-x_4)}{[\lambda^2x_4+(m^2-s)x_3+sx_3{}^2+m^2(x_1+x_2)^2+(s+m^2-\lambda^2)x_3(x_1+x_2)-tx_1x_2]^2}\,.\end{aligned} \tag{6.3}$$

[12] P. Federbush and M. Grisaru, Ann. Phys. **22**, 263, 299 (1963); J. Polkinghorne, J. Math. Phys. (to be published); R. P. Feynman (unpublished); G. Tiktopoulos, Phys. Rev. **131**, 2373 (1963).

The limit of interest in our work is large t and fixed finite s. It is clear that the only range of the parameters x_1 and x_2 which is important is $x_1 \approx 0$, $x_2 \approx 0$, otherwise $\mathfrak{M}_0$ will surely go like $1/t^2$. We may therefore set x_1 and x_2 equal to zero everywhere in the integrand except in the term $(-t)x_1x_2$ and integrate x_1 and x_2 from zero to some small positive value. After an elementary calculation we find in the limit $(-t) \to \infty$:

$$\mathfrak{M}_0 = \frac{1}{16\pi^2} \int_0^1 dx \frac{1}{\lambda^2(1-x)+(m^2-s)x+sx^2-i\epsilon} \times \left[\frac{\ln(-t)}{(-t)}\right]. \quad (6.4)$$

The usual $i\epsilon$ heretofore unwritten has been restored. The coefficient of $\ln(-t)/(-t)$ is just what we called I_0 in Sec. 5 and we recall that it has the dispersion representation

$$\frac{1}{16\pi^2} I_0 = \frac{1}{16\pi^2} \int_{(m+\lambda)^2}^{\infty} ds' \frac{1}{k's'^{1/2}} \frac{-1}{s'-s'-i\epsilon}$$

where k' is the center of mass momentum of two particles of mass m and λ with total energy $s'^{1/2}$. It is perhaps worth noting that the mass m occurring in the last two formulas is that associated with the internal propagator, $[(p-l)^2+m^2]^{-1}$. In the large t limit the masses of the other propagators as well as the external masses disappear. We are now in a position to deal with the real problem. The matrix element associated with Fig. 3, a_1 (call it $\mathfrak{M}_1$) with all the correct factors is

$$\mathfrak{M}_1 = \gamma^4 \int \frac{d^4l}{(2\pi)^4 i} \frac{1}{l^2+\lambda^2} \gamma_\lambda \frac{1}{i\gamma\cdot(p_2-l)+m} \times \Gamma_\mu \frac{1}{i\gamma\cdot(p-l)+m} \Gamma_\nu \frac{1}{i\gamma\cdot(p_1-l)+m} \gamma_\lambda, \quad (6.5)$$

where we recall that

$$\Gamma_\mu = \gamma_\mu - \gamma\cdot k_2 k_{1\mu}/k_1\cdot k_2, \quad \Gamma_\nu = \gamma_\nu - \gamma\cdot k_1 k_{2\nu}/k_1\cdot k_2.$$

Using the fact that $\mathfrak{M}_1$ is to be evaluated between $\bar{u}(p_2)$ and $u(p_1)$ we see that $\gamma_\lambda[-i\gamma\cdot(p_2-l)+m]$ on the left may be written as $[-2ip_{2\lambda}+\gamma_\lambda i\gamma\cdot l]$ and $[-i\gamma\cdot(p_1-l)+m]\gamma_\lambda$ on the right becomes $[-2ip_{1\lambda}+i\gamma\cdot l\gamma_2]$. We find then for $\mathfrak{M}_1$ the result

$$\mathfrak{M}_1 = \gamma^4 \int \frac{d^4l}{(2\pi)^4 i} \frac{[-2ip_{2\lambda}+\gamma_\lambda i\gamma\cdot l]\Gamma_\mu[-i\gamma\cdot(p-l)+m]\Gamma_\nu[-2ip_{1\lambda}+i\gamma\cdot l\gamma_\lambda]}{[p_1-l)^2+m^2][(p_2-l)^2+m^2][(p-l)^2+m^2][l^2+\lambda^2]}. \quad (6.6)$$

We introduce the Feynman parameters $x_1x_2x_3x_4$ and make the displacement $l \to l+r$, $r = p_1x_1+p_2x_2+px_3$. It is not difficult to show that the only term of $\mathfrak{M}_1$ that we need to keep is the explicit $p_1\cdot p_2 \approx t/2$ together with the displacement x_3p in the middle term. The point is that all other ways of producing a t in the numerator (needed to cancel the natural $1/t$ occurring in $\mathfrak{M}_0$) inevitably involve an x_1 or x_2, which removes the logarithm and makes the term too small to be of interest here. There results, in the large z limit,

$$\mathfrak{M}_1 = (-\gamma^4/8\pi^2) i\Gamma_\mu[(-i\gamma\cdot p+m)I_0 + i\gamma\cdot pI_1] i\Gamma_\nu \ln(-t), \quad (6.7)$$

where I_0 and I_1 are as defined in Sec. 5.

The entire answer for the matrix element, including all terms through order γ^4 in the limit of large z, is obtained by adding to $\mathfrak{M}_1$ the Born approximation:

$$\mathfrak{M} = \gamma^2 i\Gamma_\mu[1/(i\gamma\cdot p+m]\{1+(\gamma^2/8\pi^2)(-i\gamma\cdot p-m) \times[(-i\gamma\cdot p+m)I_0+i\gamma\cdot pI_1]\ln(-t)\} i\Gamma_\nu. \quad (6.8)$$

This is precisely of the predicted form (with no unexpected appendages) namely, (to fourth order),

$$\mathfrak{M} = \gamma^2 i\Gamma_\mu[(-z)^{\alpha(-i\gamma\cdot p)}/(i\gamma\cdot p+m)] i\Gamma_\nu, \quad (6.9)$$

where

$$\alpha(W) = (\gamma^2/8\pi^2)(W-m)[(W+m)I_0 - WI_1]. \quad (6.10)$$

The slight modifications caused by signature are discussed in connection with the sixth order, where they make their first appearance.

This whole calculation is really very simple, but we would like to point out that, if one does not use our external gauge, it takes on a nightmarish quality. If one simply uses "good old gauge," where vertices are written as γ_μ and γ_ν, from numerator spinology one gets powers of $(\cos\theta)^2$. As a result all diagrams which have t dependence ($\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ and $\bar{a}_4$ of Fig. 3) must be added up in detail and terms like $\ln t/t^2$ must be religiously kept. There then appear fantastic cancellations, ultimately leading to the result we obtained so easily. One particularly troublesome aspect of this straightforward approach is the cancellation of the $[\ln(-t)]^2$ term between a_1 and $\bar{a}_1$ of Fig. 3 and the need for extracting the terms proportional to $\ln(-t)$ hiding under the leading power. We mention this because the same disease plagues us in our discussion of the higher orders.

We turn now to the 6th and effectively nth-order contributions. Unfortunately our treatment of 6th order is still incomplete. Although we can isolate the expected contribution, the term proportional to $[\ln(+t)]^2$ [recall the alternation from order to order from $\ln(-t)$ to $\ln(+t)$], we have not shown the absence of other $[\ln(+t)]^2$ terms without imaginary parts in s. The external gauge which was so useful in 4th order

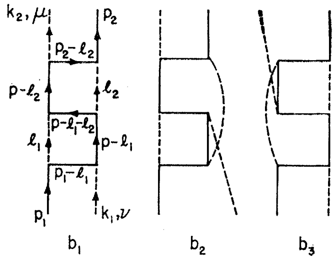

FIG. 4. Relevant diagrams for sixth-order Compton scattering.

is not sufficient to isolate the one diagram of interest and at least a subset of the total 6th order graphs must be considered together. This will become clear in a moment. It is of course true that in addition to the expected term which we anticipate being roughly of the form $[\alpha \ln t]^2/2$ with the same α as previously determined, there will be one with a three-particle threshold proportional to $\ln(-t)$ which we do not discuss here although it is not too hard to compute. It would lead to part of the γ^4 contribution to $\alpha(-i\gamma\cdot p)$. (There are also γ^4 terms with a two-particle threshold.) We are therefore going to confine our attention to the highest power of $\ln t$ for the given power γ^6, namely $(\ln t)^2$.

It is evident that it is the diagram b_1 of Fig. 4 which we expect to be the interesting one, since it is the only one with a repeated two-particle intermediate state. The role of the other two diagrams in Fig. 4 will be explained shortly. The same external gauge used previously will be taken for the external photon vertices. The matrix element becomes

$$\mathfrak{M}_1^{(6)} = \gamma^6 \int \frac{d^4 l_1}{(2\pi)^4 i} \int \frac{d^4 l_2}{(2\pi)^4 i} \gamma_\lambda \frac{1}{i\gamma\cdot(p_2-l_2)+m} \times i\Gamma_\mu \frac{1}{i\gamma\cdot(p-l_2)+m} \gamma_\sigma \frac{1}{i\gamma\cdot(p-l_1-l_2)+m} \times \gamma_\lambda \frac{1}{i\gamma\cdot(p-l_1)+m} i\Gamma_\nu \frac{1}{i\gamma\cdot(p_1-l_1)+m} \times \gamma_\sigma \frac{1}{(l_2^2+\lambda^2)} \frac{1}{(l_1^2+\lambda^2)}. \quad (6.11)$$

In order to simplify the writing, for the time being we drop the factors and deal only with the numerator which results from rationalizing the fermion propagators; call it $\mathfrak{N}_1$:

$$\mathfrak{N}_1 = \gamma_\lambda[-i\gamma\cdot(p_2-l_2)+m] i\Gamma_\mu [-i\gamma\cdot(p-l_2)+m] \times \gamma_\sigma[-i\gamma\cdot(p-l_1-l_2)+m]\gamma_\lambda[-i\gamma\cdot(p-l_1)+m] \times i\Gamma_\nu[-i\gamma\cdot(p_1-l_1)+m]\gamma_\sigma. \quad (6.12)$$

The first step in the isolation of the desired term is to note as in the fourth-order case the relations

$$\gamma_\lambda[-i\gamma\cdot(p_2-l_2)+m] = [-2ip_{2\lambda} + \gamma_\lambda i\gamma\cdot l_2],$$
$$[-i\gamma\cdot(p_1-l_1)+m]\gamma_\sigma = [-2ip_{1\sigma} + i\gamma\cdot l_1 \gamma_\sigma].$$

We now retain only the terms $(-2ip_{2\lambda})$ and $(-2ip_{1\sigma})$ as these are the ones with the greatest potential for producing an explicit factor of t in $\mathfrak{N}_1$. In fact these are the only relevant terms, but we leave it to the reader to convince himself of this. Then $\mathfrak{N}_1$ becomes

$$\mathfrak{N}_1 \approx -4i\Gamma_\mu[-i\gamma\cdot(p-l_2)+m]\gamma\cdot p_1[-i\gamma\cdot(p-l_1-l_2) + m]\gamma\cdot p_2[-i\gamma\cdot(p-l_1)+m]i\Gamma_\nu. \quad (6.13)$$

Next we move $\gamma\cdot p_2$ to the left and $\gamma\cdot p_1$ to the right so that they may ultimately act on the appropriate spinors to give (im). We make use of the fact that the only important components of l_1 and l_2 are parallel to p in the limit of large t; components parallel to p_1 and p_2 will bring in Feynman parameters which will make the integral too small to be of interest. Further, only $\{\gamma\cdot p_1, \gamma\cdot p_2\}$ is of significant size, since $\{\gamma\cdot p_1, \gamma\cdot p\} = \{\gamma\cdot p_2, \gamma\cdot p\} = -(s+m^2-\lambda^2)$ and $\{\gamma\cdot p_1, \gamma\cdot p_2\} \approx t/2$. We are left with

$$\begin{aligned}\mathfrak{N}_1 \approx & -4(t/2)i\Gamma_\mu[-i\gamma\cdot(p-l_2)+m] \\ & \times[+i\gamma\cdot(p-l_1-l_2)+m][-i\gamma\cdot(p-l_1)+m]i\Gamma_\nu, \\ = & -4(t/2)i\Gamma_\mu[-i\gamma\cdot(p-l_2)+m] \\ & \times[-i\gamma\cdot p-m][-i\gamma\cdot(p-l_1)+m]i\Gamma_\nu \qquad (6.14)\\ & -4(t/2)i\Gamma_\mu[(p-l_2)^2+m^2][-i\gamma\cdot(p-l_1)+m]i\Gamma_\nu \\ & -4(t/2)i\Gamma_\mu[-i\gamma\cdot(p-l_2)+m][(p-l_1)^2+m^2]i\Gamma_\nu.\end{aligned}$$

The second form comes from writing

$$[i\gamma\cdot(p-l_1-l_2)+m] = [i\gamma\cdot(p-l_2)+m+i\gamma\cdot(p-l_1)+m-i\gamma\cdot p-m].$$

The first term in this equation for $\mathfrak{N}_1$ is, as we show in a moment, just what we want. The remaining two terms are unfortunately quite disgusting. They give rise to a large t dependence of the form $t(\ln t)^3$. It has been shown by P. Federbush that these terms are cancelled by the diagrams b_2 and b_3 of Fig. 4. In fact the result for $\mathfrak{M}_1^{(6)}$ implied by the first term in the above formula for $\mathfrak{N}_1$ is the total answer in so far as the absorptive part of the amplitude in the s channel is concerned. The only remaining task in 6th order is to show that the sum of the $(\ln t)^2$ terms which have constant or polynomial (in s) coefficients indeed cancel. This has not yet been completed. It is unfortunate that the use of our external gauge, which even in 6th order serves to easily eliminate many diagrams, does not at the same time dispose of the $(\ln t)^3$ terms just as $(\ln t)^2$ terms disappeared in 4th order. We were unable to find any choice of gauge that would do the job. It is the presence of these extraneous contributions (i.e., higher powers of $\ln t$) which makes it useless to write down an integral equation for the ladders of the variety shown in Fig. 3, a_1 and Fig. 4, b_1, and solve for the part of α which has the two particle cut, to all orders of γ^2.

It is now quite easy to see that the term we have isolated in 6th order fits into the predicted pattern. Quite explicitly we have

$$\mathfrak{M}_1^{(6)}=-4\gamma^6\left(\frac{t}{2}\right)\int\frac{d^4l_1}{(2\pi)^4i}\int\frac{d^4l_2}{(2\pi)^4i}\frac{i\Gamma_\mu[-i\gamma\cdot(p-l_2)+m[-i\gamma\cdot p-m][-i\gamma\cdot(p-l_1)+m]i\Gamma_\nu}{D}, \tag{6.15}$$

where

$$D=[(p_2-l_2)^2+m^2][(p-l_2)^2+m^2][(p-l_1-l_2)^2+m^2][(p-l_2)^2+m^2][(p_2-l_2)^2+m^2][l_2^2+\lambda^2][l_1^2+\lambda^2].$$

This integral can be discussed by the methods of Federbush and Grisaru[12]; except for the numerators they have actually done it. It turns out that the factoring of the integral into a product obtains here also and one simply replaces the numerator l_1 and l_2 by their components parallel to p (times the appropriate Feynman integration parameter). The result is

$$\mathfrak{M}_1^{(6)}=-(\gamma^2/2)i\Gamma_\mu(-i\gamma\cdot p-m)\{(\gamma^2/8\pi^2)\times[(-i\gamma\cdot p+m)I_0+i\gamma\cdot pI_1]\}^2 i\Gamma_\nu(\ln t)^2. \tag{6.16}$$

This is exactly what is predicted from the formula

$$\mathfrak{M}=(\gamma^2/2)i\Gamma_\mu\{[(-t)^{\alpha(-i\gamma p)}+(t)^{\alpha(-i\gamma\cdot p)}]-[(-t)^{-\alpha(-i\gamma\cdot p)}-(t)^{-\alpha(-i\gamma\cdot p)}]\times[1/(i\gamma\cdot p+m)]i\Gamma_\nu. \tag{6.17}$$

The formula now holds through order γ^6 subject to the previously mentioned qualifications. It is perhaps worth remarking that the explicit appearance of the factor $(-i\gamma\cdot p-m)$ in 6th order is an entirely nontrivial miracle; we see in the following paper that the corresponding factor does not occur for a scalar target particle. It is this factor which shows that the trajectory is passing through the elementary particle pole.

It should be quite apparent that the isolation of the term obtained by expanding the above formula to arbitrary order can be done exactly as we did for the 6th order. Needless to say, the cancellation of unwanted powers of $\ln t$ becomes more and more of a problem in higher orders.

7. CONCLUSIONS

We have examined the scattering of neutral vector mesons from spinor nucleons in perturbation theory, keeping the highest power of $\ln z$ in each order of the coupling constant γ^2. Except for checking that subtraction constants vanish in sixth and higher orders, we have verified that the nucleon lies on a Regge trajectory in this approximation, with $\alpha(W)$ given by a power series in γ^2, of which we have calculated the first term. To carry the investigation that far, it was not necessary to go beyond elastic unitarity for the scattering.

The crucial features of the theory that allow the nucleon to turn into a Regge pole as a result of radiative corrections are:

(a) the existence of at least one "nonsense" channel at $J=1/2$ that couples to the "sense" channels in which the nucleon appears as an intermediate state;
(b) the factoring of the Born approximation.

We have shown above, as in II, that these conditions are necessary and that apart from possible subtraction constants in higher order, they are sufficient in our approximation.

In a field theory of just scalar or pseudoscalar mesons and spinor nucleons (or scalar mesons and scalar nucleons), the two-particle channels that communicate with the nucleon do not include a nonsense channel at the angular momentum of the nucleon. The mathematical condition for the appearance of such a channel is that

$$J\leqslant s_1+s_2-1,$$

where J is the spin of the particle that is to lie on a trajectory and s_1 and s_2 are the spins of the particles into which it dissociates.

It is, of course, possible that the particle may turn into a Regge pole as a result of dissociation into more than two particles, but if so we conjecture that the condition is

$$J\leqslant s_1+s_2+s_3-2,\quad \text{etc.}$$

If that is right, then in the absence of vector mesons the nucleon, introduced as an "elementary" particle into a renormalized field theory, will not lie on a Regge trajectory.

In I, the question was also raised of whether a *scalar* nucleon, coupled through a conserved current to a vector meson, would lie on a Regge trajectory. Here a two-particle nonsense channel is available, but the Born approximation does not factor. The details are given in the next paper, where it is shown that a Regge trajectory develops, but the scalar nucleon does not lie on it. (The interesting possibility is raised of mutilating the theory so that the Born approximation does factor; if the mutilated theory can be made finite, then the Regge trajectory does pass through the nucleon.[3])

The theory of spinor and vector particles (quantum electrodynamics with massive photons) thus seems to have special virtues. It is possible, as mentioned in I, that the massive photon also lies on a trajectory when coupling to three-photon states is included, but that question is not yet settled. The same theory may also give rise to a Pomeranchuk trajectory (passing through $J=1$ at zero energy); this possibility was raised by Freund and Oehme[13] and some of the present authors[14] have extended this work. It is certainly worthwhile to pay further attention to the vector-spinor field theory. We mention here some points that seem particularly interesting:

(a) The limiting case of true quantum electrodynamics ($\lambda\to 0$) should be studied. Here the individual scattering amplitudes actually vanish, because of possibility of radiating an infinite number of soft photons. One might, however, factor out of the two-particle amplitudes the main λ dependence, then allow λ to approach zero, and finally consider large $\cos\theta$. This type of double limit gives the physically interesting behavior at high energies in the crossed reaction; it is not the same as taking our results at large $\cos\theta$ and then considering small λ.

(b) Possible applications of neutral vector meson theory to strong interactions cannot be excluded. There is, however, a difficulty if the resulting description of strongly interacting particles is to resemble the eightfold way. If we introduce eight degenerate spin 1/2 baryons coupled through the conserved baryon current to a neutral vector meson, the resulting theory is symmetrical under the group $SU(8)$, with 63 generators. It is hard to see how to reduce the symmetry to $SU(3)$ in a natural way without, for example, introducing a further octet of vector mesons described by a theory of the Yang-Mills type; such a theory does not seem to be renormalizable in the usual sense and we have no evidence that it is consistent with Regge pole behavior. Of course we can eventually introduce mass differences among the eight baryons and/or coupling of the original vector meson to the strangeness current as well as the baryon current, but such terms break the symmetry rather than reducing it from $SU(8)$ to $SU(3)$.

(c) Ignoring the difficulties we have just mentioned, we may speculate about the relation of neutral vector meson theory to a theory of strong interactions based on dispersion relations with only moving singularities in the J plane. Chew and Frautschi[15] have raised the hope that in the absence of fixed singularities in J (for $\mathrm{Re}J>0$) all coupling constants and mass ratios may be determined by the "bootstrap" mechanism. In our field theory it is possible that there are only moving singularities, but the coupling constant γ^2 and the mass ratio λ/m are introduced arbitrarily. Is there some kind of consistency that determines their values? If not, and if Chew and Frautschi are right, then is there some experimental distinction between a Regge trajectory determined dynamically by a bootstrap and one that arises from an elementary particle of field theory?

(d) If our mathematical nucleon has anything to do with real nucleons, there may be some importance to the trajectory of opposite signature which has, in second order, a value of α equal and opposite to that of the nucleon. Could this trajectory have any connection with the "second resonance" around 1510 MeV, which is thought to have $J=3/2^-$?

(e) In our approximation, we have not treated low enough powers of $\ln z$ for a given order in γ^2 to encounter the Gribov phenomenon[16] or the cuts in the angular momentum plane that Mandelstam has found and that are supposed to exclude Gribov's essential singularities from the physical sheet. It will be instructive to see how these things go in the vector-spinor field theory.

(f) It is interesting that the Regge behavior of the nucleon in vector meson theory persists even if we study scattering by nucleons of scalar or pseudoscalar mesons, with these particles introduced only as external lines. In I (particularly the Erratum) this problem has been discussed, but we do not fully understand the meaning of the result. Mathematically, the basis is the factoring, in Born approximation, of all the scattering amplitudes with vector, scalar, and pseudoscalar mesons coming or going out.

Finally, let us re-emphasize our belief that additional experiments in the laboratory of Feynman diagrams will be of great value to all students of relativistic quantum mechanics, including those who use the language of "S-matrix theory" and those who are investigating the consequences of the "axioms."

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APPENDIX A

Properties of e and c Functions

We refer to a list of properties of $d_{\lambda\mu}{}^J(\theta)$ given by Jacob and Wick[4] in their Appendix A. Using our definitions (2.8) and (2.10) we find the corresponding relations for $e_{\lambda\mu}{}^{J\pm}(z)$ and $c_{\lambda\mu}{}^{J\pm}(z)$.

[13] P. Freund and R. Oehme, Phys. Rev. Letters **10**, 199, 315 (1963).

[14] M. Gell-Mann, M. L. Goldberger, and F. E. Low (to be published).

[15] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **8**, 41 (1962).

[16] V. Gribov, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962).

The functions need be given only for positive subscripts in a particular order. We may then find $e^{J\pm}$ and $c^{J\pm}$ for other values of the subscripts from the relations

$$e_{-\mu,-\lambda}{}^{J\pm}(z)=e_{\lambda\mu}{}^{J\pm}(z)\,, \tag{A1}$$

$$e_{\mu\lambda}{}^{J\pm}(z)=(-1)^{\lambda-\mu}e_{\lambda\mu}{}^{J\pm}(z)\,, \tag{A2}$$

$$e_{\lambda,-\mu}{}^{J\pm}(z)=\pm(-1)^{\lambda+\lambda_m}e_{\lambda\mu}{}^{J\pm}(z)\,, \tag{A3}$$

with corresponding formulas for the c's.

$$[\lambda_m=\max(|\lambda|,|\mu|)].$$

Under $\theta\to\pi-\theta$, we have $z\to -z$ and we find

$$e_{\lambda\mu}{}^{J\pm}(-z)=\pm(-1)^{J-\lambda_m}e_{\lambda\mu}{}^{J\pm}(z) \tag{A4}$$

and likewise for the c's. In fact, if we define

$$e_{\lambda\mu}{}^{J}(z)\equiv e_{\lambda\mu}{}^{J+}(z)+e_{\lambda\mu}{}^{J-}(z)=[\sqrt{2}\cos(\theta/2)]^{-|\lambda+\mu|}\times[\sqrt{2}\sin(\theta/2)]^{-|\lambda-\mu|}d_{\lambda\mu}{}^{J}(\theta)\,, \tag{A5}$$

$$c_{\lambda\mu}{}^{J}(z)\equiv c_{\lambda\mu}{}^{J+}(z)+c_{\lambda\mu}{}^{J-}(z)=[\sqrt{2}\cos(\theta/2)]^{|\lambda+\mu|}\times[\sqrt{2}\sin(\theta/2)]^{|\lambda-\mu|}d_{\lambda\mu}{}^{J}(\theta)\,, \tag{A6}$$

then we may evaluate $e_{\lambda\mu}{}^{J\pm}$ and $c_{\lambda\mu}{}^{J\pm}$ as the parts of $e_{\lambda\mu}{}^{J}$ and $c_{\lambda\mu}{}^{J}$, respectively, that transform according to (A4) under $z\to -z$. The functions $e_{\lambda\mu}{}^{J}(z)$ are proportional to the so-called Jacobi polynomials

$$P_{J-\lambda}{}^{\lambda-\mu,\lambda+\mu}(z)\quad\text{for}\quad \lambda\geqslant|\mu|\,.$$

The recursion relations for $d_{\lambda\mu}{}^{J}$ given by Jacob and Wick are easily transformed into recursion relations for $e_{\lambda\mu}{}^{J}$ and $c_{\lambda\mu}{}^{J}$. Let us consider positive indices only. When $\mu\geqslant\lambda$ we obtain

$$e_{\lambda\;\mu+1}{}^{J}=[(J+\mu+1)(J-\mu)]^{1/2}De_{\lambda\mu}{}^{J}\,, \tag{A7}$$

where D means d/dz. For $\mu<\lambda$, we find

$$e_{\lambda\;\mu+1}{}^{J}=[(J+\mu+1)(J-\mu)]^{1/2}\times[(1-z)D+\mu-\lambda]e_{\lambda\mu}{}^{J}\,. \tag{A8}$$

Using (A2), we can now derive an exact general expression for integral J and non-negative λ, μ

$$e_{\lambda\mu}{}^{J}=(-1)^{\lambda}[(J-\lambda)!(J-\mu)!/(J+\lambda)!(J+\mu)!]^{1/2}\times D^{|\mu-\lambda|}(D^2-D-zD^2)^{m}P_J\,, \tag{A9}$$

where we have used the fact that $e_{00}{}^{J}=P_J$. Here $m=\min(\lambda,\mu)$. It is clear that at large values of z, the larger of $e_{\lambda\mu}{}^{J+}$ and $e_{\lambda\mu}{}^{J-}$ goes like $D^{\lambda_m}P_J$ and transforms under $z\to -z$ with a factor $(-1)^{J-\lambda_m}$; thus it is $e_{\lambda\mu}{}^{J+}$ that dominates.

For half-integral J we find for positive λ and μ

$$e_{\lambda\mu}{}^{J}=(-1)^{\lambda-1/2}(J+1/2)\times[(J-\lambda)!(J-\mu)!/(J+\lambda)!(J+\mu)!]^{1/2}\times D^{|\mu-\lambda|}(D^2-D-zD^2)^{m-1/2}e_{1/2\;1/2}{}^{J}\,, \tag{A10}$$

with $e_{1/2\;1/2}{}^{J}=2^{-1/2}(J+1/2)^{-1}(P_{J+1/2}'-P_{J-1/2}')$ from Jacob and Wick. Again the $+$ function dominates at large z.

Using (A9) and (A10) we generate the following list of e functions for non-negative λ and μ up to 2:

$$e_{00}{}^{J+}=P_J\,,\qquad e_{00}{}^{J-}=0\,.$$

$$-e_{10}{}^{J+}=e_{01}{}^{J-}=P_J'/[J(J+1)]^{1/2}\,,\qquad -e_{10}{}^{J-}=e_{01}{}^{J-}=0\,.$$

$$e_{20}{}^{J+}=e_{02}{}^{J+}=P_J''/[(J-1)J(J+1)(J+2)]^{1/2}\,,\qquad e_{20}{}^{J-}=e_{02}{}^{J-}=0\,.$$

$$e_{11}{}^{J+}=(P_J'+zP_J'')/J(J+1)\,,\qquad e_{11}{}^{J-}=-P_J''/J(J+1)\,.$$

$$-e_{21}{}^{J+}=e_{12}{}^{J+}=\frac{2P_J''+zP_J'''}{J(J+1)[(J-1)(J+2)]^{1/2}}\,,\qquad -e_{21}{}^{J-}=e_{12}{}^{J-}=\frac{-P_J'''}{J(J+1)[(J-1)(J+2)]^{1/2}}\,.$$

$$e_{22}{}^{J+}=\frac{2P_J''+4zP_J'''+(z^2+1)P_J{}^{\text{iv}}}{(J-1)J(J+1)(J+2)}\,,\qquad e_{22}{}^{J-}=\frac{-4P_J'''-2zP_J{}^{\text{iv}}}{(J-1)J(J+1)(J+2)}\,.$$

$$e_{1/2\;1/2}{}^{J+}=(1/\sqrt{2})[P_{l+1}'/(l+1)]\,,\qquad e_{1/2\;1/2}{}^{J-}=(-1/\sqrt{2})[P_l'/(l+1)]\,.$$

$$-e_{3/2\;1/2}{}^{J+}=e_{1/2\;3/2}{}^{J+}=\frac{1}{\sqrt{2}}\frac{P_{l+1}''}{(l+1)[l(l+2)]^{1/2}}\,,\qquad -e_{3/2\;1/2}{}^{J-}=e_{1/2\;3/2}{}^{J-}=\frac{-1}{\sqrt{2}}\frac{P_l'}{(l+1)[l(l+2)]^{1/2}}\,.$$

$$e_{3/2\;3/2}{}^{J+}=\frac{1}{\sqrt{2}}\frac{P_{l+1}''+zP_{l+1}'''+P_l'''}{l(l+1)(l+2)}\,,\qquad e_{3/2\;3/2}{}^{J-}=\frac{-1}{\sqrt{2}}\frac{P_{l+1}'''+zP_l'''+P_l''}{l(l+1)(l+2)}\,. \tag{A11}$$

Now let us discuss the calculation of c functions. By definition of $c_{\lambda\mu}{}^{J}$ and $e_{\lambda\mu}{}^{J}$ we have, of course,

$$c_{\lambda\mu}{}^{J}=(1+z)^{|\lambda+\mu|}(1-z)^{|\lambda-\mu|}e_{\lambda\mu}{}^{J} \tag{A12}$$

and this formula provides a way of finding the c's from the e's; and the $\pm$ parts can be picked out by their behavior under $z\to -z$. It is more convenient, however, to express $c_{\lambda\mu}{}^{J}$ as a linear combination, with constant coefficients, of Legendre functions ranging from $P_{J-\lambda_m}$ to $P_{J+\lambda_m}$. We start with $c_{00}{}^J=P_J$ and $c_{1/2\ 1/2}{}^J=(1+z)e_{1/2\ 1/2}{}^J=(1/\sqrt{2})\times(P_l+P_{l+1})$.

We define the operators L, Δ_+, and Δ_- by the relations $LP_n\equiv nP_n$, $\Delta_+P_n\equiv P_{n+1}$, and $\Delta_-P_n\equiv P_{n-1}$. Then the recursion relations of Jacob and Wick give, for non-negative λ and μ, the results

$$c_{\lambda+1,\lambda+1}{}^J=-\frac{1}{(J+\lambda+1)(J-\lambda)}\left\{4\lambda(\lambda+1)-L(L+1)-\Delta_+\frac{L+1}{2L+1}(L-2\lambda)^2-\Delta_-\frac{L}{2L+1}(L+1-2\lambda)^2\right\}c_{\lambda\lambda}{}^J, \quad \text{(A13)}$$

$$\mu\geqslant\lambda:\ c_{\lambda,\mu+1}{}^J=\frac{1}{[(J+\mu+1)(J-\mu)]^{1/2}}\left\{(\Delta_--\Delta_+)\frac{L(L+1)}{2L+1}-2\lambda+2\mu\left(\Delta_+\frac{L+1}{2L+1}+\Delta_-\frac{L}{2L+1}\right)\right\}c_{\lambda\mu}{}^J. \quad \text{(A14)}$$

By using these formulas we can obtain all the c functions, step by step, in the desired form. We thus generate the following list, for non-negative λ and μ up to 2:

$$\begin{aligned}
&c_{00}{}^{J+}=P_J. \qquad\qquad c_{00}{}^{J-}=0.\\
&-c_{10}{}^{J+}=c_{01}{}^{J+}=[J(J+1)]^{1/2}/2J+1(P_{J-1}-P_{J+1}), \qquad -c_{10}{}^{J-}=c_{01}{}^{J}=0.\\
&c_{20}{}^{J+}=c_{02}{}^{J+}=\frac{[(J-1)J(J+1)(J+2)]^{1/2}}{(2J-1)(2J+1)(2J+3)}[(2J+3)P_{J-2}-2(2J+1)P_J+(2J-1)P_{J+2}],\\
&\qquad c_{02}{}^{J-}=c_{20}{}^{J-}=0.\\
&c_{11}{}^{J+}=[(J+1)P_{J-1}+JP_{J+1}]/(2J+1), \qquad c_{11}{}^{J-}=P_J.\\
&-c_{21}{}^{J+}=c_{12}{}^{J+}=\frac{[(J-1)(J+2)]^{1/2}}{(2J-1)(2J+1)(2J+3)}[(J+1)(2J+3)P_{J-2}-3(2J+1)P_J-J(2J-1)P_{J+2}],\\
&\qquad -c_{21}{}^{J-}=c_{12}{}^{J-}=\frac{[(J-1)(J+2)]^{1/2}}{2J+1}(P_{J-1}-P_{J+1}).\\
&c_{22}{}^{J+}=\frac{(J+1)(J+2)(2J+3)P_{J-2}+6(J-1)(J+2)(2J+1)P_J+(J-1)J(2J-1)P_{J+2}}{(2J-1)(2J+1)(2J+3)}, \qquad \text{(A15)}\\
&\qquad c_{22}{}^{J-}=\frac{2(J+2)P_{J-1}+2(J-1)P_{J+1}}{2J+1}.\\
&c_{1/2\ 1/2}{}^{J+}=(1/\sqrt{2})P_l, \qquad c_{1/2\ 1/2}{}^{J-}=(1/\sqrt{2})P_{l+1}.\\
&-c_{3/2\ 1/2}{}^{J+}=c_{1/2\ 3/2}{}^{J+}=\frac{[l(l+2)]^{1/2}}{\sqrt{2}}\frac{P_{l-1}-P_{l+1}}{2l+1}, \qquad -c_{3/2\ 1/2}{}^{J-}=c_{1/2\ 3/2}{}^{J-}=\frac{[l(l+2)]^{1/2}}{\sqrt{2}}\frac{P_l-P_{l+2}}{2l+3}\\
&c_{3/2\ 3/2}{}^{J+}=\frac{1}{\sqrt{2}}\frac{(l+2)P_{l-1}+3lP_{l+1}}{2l+1}, \qquad c_{3/2\ 3/2}{}^{J-}=\frac{1}{\sqrt{2}}\frac{3(l+2)P_l+lP_{l+2}}{2l+3}.
\end{aligned}$$

APPENDIX B

Reggeizing, Sense and Nonsense, Compensating Trajectories

We present here an expanded discussion of Reggeization in the presence of spin as treated in Ref. 6, with emphasis on sense and nonsense and on compensating trajectories, such as the P and Q trajectories.[6] The notion of compensating trajectories has been attacked by Berestetsky,[7] but there is no basis for his criticisms, as can be seen below.

For simplicity, we ignore exchange forces and signature until the very end. In Reggeizing, we do not prove that the contribution of the large semicircular contour can be discarded as it recedes to infinity. (In our work on vector-spinor scattering in perturbation theory, the agreement between the asymptotic behavior of the f's and the behavior of the F's in the l plane shows that we do not have to worry about trouble from the large contour, at least to the right of $l=0$.[17])

A more serious simplication is our neglect of complications arising from the third Mandelstam weight function, for example the Gribov phenomenon[16] and the associated cuts in the l plane discussed by Mandlestam.[18] We ignore cuts and essential singularities in

[17] We wish to thank Professor S. Coleman for this comment.
[18] S. Mandelstam (to be published).

the J plane and assume the absence of fixed poles. That corresponds to the case of coupled Schrödinger equations or to field theory with the approximations used in this paper.

Our discussion lacks rigor in one further respect. We use the Froissart definitions of the F^J for general J in terms of integrals of Q functions times the weight functions w of the f amplitudes; but these definitions are strictly correct only for sufficiently large ReJ. At other values of J, they must be analytically continued. We treat these definitions, however, as if they applied at all values of J that we consider; our manipulations of these definitions must be regarded as heuristic.

Let us now review Reggeization in the spinless case, using essentially the method of Mandelstam.[19] We start with a partial wave expansion

$$f(z)=\sum_{J=0}^{\infty}(2J+1)P_J(z)F^J \tag{B1}$$

and a Froissart inversion formula

$$F^J=\int dz Q_J(z)w(z)\,, \tag{B2}$$

where w is the weight function of f in a dispersion relation in z. We avoid fixed poles at the negative integers. (They are in fact absent for the Schrödinger equation, for example, although they seem to be there in the lowest Born approximation. When higher approximations are included, these singularities become moving poles.) In our heuristic language, that corresponds to setting

$$\int dz P_J(z)w(z)=0\,,\quad J=0,1,2,\cdots \tag{B3}$$

(at all but isolated values of the energy), since at the negative integers the general relation

$$Q_J-Q_{-J-1}=\pi\cot J\pi P_J \tag{B4}$$

yields

$$Q_J\approx\pi\cot J\pi P_{-J-1}\,.$$

Now we use (B4) to express P_J in the form

$$P_J(z)=\mathcal{P}_J(z)+\mathcal{P}_{-J-1}(z)\,,$$

where

$$\mathcal{P}_J\equiv-Q_{-J-1}\pi^{-1}\tan J\pi=\Gamma(J+1/2)[\Gamma(J+1)]^{-1}\pi^{-1/2}\times(2z)^J F(-J/2,1/2-J/2;1/2-J;1/z^2) \tag{B5}$$

as in Ref. 6. At $J=0,1,2,\cdots$, we have $\mathcal{P}_J=P_J$, while at negative integral J we have $\mathcal{P}_J=0$. Since F^J is finite at negative integral J, we can extend the sum in (B1) over all the integers if we write

$$f(z)=\sum_{J=-\infty}^{J=+\infty}(2J+1)\mathcal{P}_J(z)F^J\,. \tag{B6}$$

Now we convert to a contour integral, with contour enclosing the whole real axis,

$$f(z)=\frac{1}{2\pi i}\oint\frac{(2J+1)}{\sin\pi J}\mathcal{P}_J(-z)F^J\,. \tag{B7}$$

Actually $\mathcal{P}_J$ has poles at the half-integers, but the residues cancel in the following way: the one at $J=-1/2$ is canceled by the factor $2J+1$; the others cancel in pairs J, $-J-1$ of half-integers. From (B2) and (B4), we have

$$F^J=F^{-J-1}\,,\quad J\text{ half-integral}\,, \tag{B8}$$

while the residues of $(2J+1)\pi(\sin\pi J)^{-1}\mathcal{P}_J(-z)$ at half-integral J and $-J-1$ are equal and opposite.

Now we expand the contour in (B7) to infinity, throwing away everything but the part at the far left. We assume for our present purposes that only moving poles are picked up. A pole in F^J of the form $\beta(J-\alpha)^{-1}$ then gives a contribution to $f(z)$ equal to

$$-[(2\alpha+1)\pi/\sin\pi\alpha]\mathcal{P}_\alpha(-z)\beta\,. \tag{B9}$$

Suppose at a given energy a trajectory passes through a value $\alpha=J_0$, with $J_0=1/2,3/2,\cdots$, etc. Then if the residue does not vanish at that point we see from (B8) that another trajectory α' must pass through $-J_0-1$ at the same energy, with the singularities related

$$\beta/(J_0-\alpha)=[\beta'/(-J_0-1-\alpha')]+\text{nonsingular terms.} \tag{B10}$$

The singularity in (B9), at the energy for which $\alpha=J_0$, is thus compensated by an equal and opposite singularity arising from the primed trajectory passing through $-J_0-1$ at the same energy.

Now let us treat an example with spin, namely the case in which $\lambda=\mu=1$. We take $c_{11}{}^{J\pm}$ and $e_{11}{}^{J\pm}$ from Appendix A. In general, we define $C_{\lambda\mu}{}^{J\pm}$ in terms of $c_{\lambda\mu}{}^{J\pm}$ by replacing P's by Q's and we define $E_{\lambda\mu}{}^{J\pm}$ in terms of $e_{\lambda\mu}{}^{J\pm}$ by replacing P's by $\mathcal{P}$'s. We then obtain

$$f^{\pm}=\sum_{J=1}^{\infty}(2J+1)(E_{11}{}^{J+}F^{J\pm}+E_{11}{}^{J-}F^{J\mp})\,, \tag{B11}$$

$$F^{J\pm}=\int dz(W^{\pm}C_{11}{}^{J+}+W^{\mp}C_{11}{}^{J-})\,, \tag{B12}$$

with

$$E_{11}{}^{J+}=\frac{\mathcal{P}_J'+z\mathcal{P}_J''}{J(J+1)}\,,\quad E_{11}{}^{J-}=\frac{-\mathcal{P}_J''}{J(J+1)}\,, \tag{B13}$$

$$C_{11}{}^{J+}=\frac{(J+1)Q_{J-1}+JQ_{J+1}}{2J+1}\,,\quad C_{11}{}^{J-}=Q_J\,. \tag{B14}$$

[19] S. Mandelstam, Ann. Phys. 19, 254 (1962).

We have used the fact that for $J=1, 2, 3, \cdots$ we have $e_{11}{}^{J\pm}=E_{11}{}^{J\pm}$.

Again we avoid fixed poles by putting

$$\int P_n W^{\pm} dz=0, \quad n=0, 1, 2, \cdots. \tag{B15}$$

For $J=-2, -3, \cdots$ we then have $F^{J\pm}$ finite and $E_{11}{}^{J\pm}=0$, so that the partial wave sum (B11) can be extended to these values of J. For $J=0$ and -1, however, both $F^{J\pm}$ and $E_{11}{}^{J\pm}$ are finite and so it is not obvious that these values can be included in the sum. That is the point that bothered Berestetsky. We note that in this example $J=0$ is the only "nonsense" value of J (in general, for integral λ or μ we define a nonsense value to be a non-negative integer less than $|\lambda|$ or $|\mu|$). The companion point in the J plane associated by reflection through $J=-1/2$ is the point $J=-1$.

Now we can easily verify that

$$F^{0\pm}=F^{(-1)\mp} = \int dz W^{\pm}\left[Q_0-\left(\frac{d}{dJ}P_J\right)_{J=0}\right]+\int dz W^{\mp} Q_0 \tag{B16}$$

and that

$$E_{11}{}^{0\pm}=E_{11}{}^{(-1)\mp}. \tag{B17}$$

Thus we can add the values $J=0$ and $J=-1$ to the partial wave sum; they cancel.

We can now connect the partial wave sum

$$f^{\pm}=\sum_{J=-\infty}^{J=+\infty}(2J+1)(E_{11}{}^{J+}F^{J\pm}+E_{11}{}^{J-}F^{J\mp}) \tag{B18}$$

to a contour integral. At half-integral values of J we have $F^{J\pm}=F^{(-J-1)\pm}$ (note the parity index is the same on both sides) and the residue of $(2J+1)\pi(\sin\pi J)^{-1}E_{11}{}^{J\pm}$ cancels against the residue at $-J-1$ for the half-integers.

We can then expand the contour and pick up Regge pole contributions

$$\begin{aligned} f^{\pm}\approx &[(2\alpha_{\pm}+1)/\sin\pi\alpha_{\pm}]E_{11}{}^{(\alpha\pm)+}(-z)\beta_{\pm} \\ &-[(2\alpha_{\mp}+1)/\sin\pi\alpha_{\mp}]E_{11}{}^{(\alpha\mp)-}(-z)\beta_{\mp} \end{aligned} \tag{B19}$$

from trajectories α_+ and α_- corresponding to poles in F^{J+} and F^{J-}, respectively. Again we have compensation of trajectories at the half-integers, without change of parity index.

A new type of compensation has now appeared, however, the kind discussed in Ref. 6 for the P and Q trajectories. Consider a trajectory $\alpha_{\pm}$ passing through $J=0$ at a certain value of energy. If the trajectory chooses *sense* at $J=0$, then the residue $\beta_{\pm}$ for $\lambda=1$, $\mu=1$, vanishes like $\alpha_{\pm}$ as $\alpha_{\pm}\to 0$. No singularity then appears in (B19) as $\alpha_{\pm}\to 0$, which is entirely appropriate for a nonsense value of α. If the trajectory chooses *nonsense* at $J=0$, then the residue $\beta_{\pm}$ approaches a finite constant as $\alpha_{\pm}\to 0$ in our nonsense $\leftrightarrow$ nonsense amplitude. The contribution to (B19) of the trajectory then *does* have a singularity at $\alpha_{\pm}=0$, but this must be canceled by something, since the f's cannot have an actual singularity at a nonsensical value of $\alpha_{\pm}$.

The cancellation occurs through the existence of a compensating trajectory $\alpha_{\mp}$ (with opposite parity index) that passes through $J=-1$ at the same energy for which $\alpha_{\pm}$ passes through 0. We can deduce the existence of the second trajectory from (B16) and moreover we conclude that

$$\beta_{\pm}/\alpha_{\pm}=[\beta_{\mp}/(-1-\alpha_{\mp})]+\text{nonsingular terms}, \tag{B20}$$

near $\alpha_{\pm}=0$, $\alpha_{\mp}=-1$. Using the relation (B17), we see immediately that the singularities in the Regge pole contributions (B19) do compensate each other.

It is interesting to remark that the leading term at large z in $E_{11}{}^{J+}$, which goes like z^{J-1}, actually vanishes as $J\to 0$, so that the largest nonvanishing term in $E_{11}{}^{0+}$ goes like z^{-3}; thus compensation by $E_{11}{}^{(-1)-}$, which goes like z^{-3} at large z, is possible.

Our example is now easily generalized to any integral values of λ and μ. The values J_0 of J [$J_0=0, 1, \cdots\min(|\lambda|, |\mu|)-1$] for which we are dealing with a nonsense-nonsense transition compensate the corresponding terms with $J=-J_0-1$. The leading terms in $E_{\lambda\mu}{}^{J+}$ at large z vanish at $J=J_0$, down to a term of the right behavior to compensate $E_{\lambda\mu}{}^{(-J_0-1)-}$, and we have

$$E_{\lambda\mu}{}^{J_0\pm}=E_{\lambda\mu}{}^{(-J_0-1)\mp}, \quad F^{J_0\pm}=F^{(-J_0-1)\mp} \tag{B21}$$

for all relevant values of J_0.

Finally we go to half-integral values of λ and μ as in our problem of vector-spinor scattering. The only change is that compensation between all pairs of integral J now occurs with a change of parity index, while the compensation between J_0 and $-J_0-1$ [for $J_0=1/2, 3/2, \cdots\min(|\lambda|, |\mu|)-1$] occurs with no change of parity index. The J_0 values are again defined as those for which we have a nonsense-nonsense transition.